A non-trivial mixing of electromagnetic and gravitational forces[1]

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The minimal coupling principle is revisited in a group-cohomological setting. Promoting space-time translations to a "local" symmetry of the Lagrangian of the free particle once the "kinematical" symmetry group (either Galilei or Poincaré) has been centrally extended by $U(1)$ results in a new electromagnetic force of pure gravitational origin. This constitutes a preliminary attempt to a non-trivial mixing of space-time and internal gauge symmetries and/or interactions.

In standard Lagrangian formalism, promoting a given underlying rigid symmetry to "local" requires the introduction of a connection which is eventually interpreted as a potential providing the corresponding gauge interaction. This is essentially the formulation of the so-called Minimal Coupling Principle, which culminates in Utiyama's theorem [1]. Internal gauge invariance had originally led successfully to electromagnetic interaction associated with $U(1)$, then to Yang-Mills associated with isospin $SU(2)$ (valid only at the very strong limit), electroweak with $SU(2)\otimes U(1)$, and finally to strong interaction associated with colour $SU(3)$. And, more recently, there have been attempts to unify all of these into gauge groups such as $SU(5)$. On the other hand, the "local" invariance under external (space-time) symmetries, such as a subgroup of the Poincaré group, has been used to provide a gauge framework for gravity [2], although fully disconnected from the other (internal) interactions. In fact, a unification of gravity and the other interactions would have required the non-trivial mixing of the space-time group and some internal symmetry, a task explicitely forbidden by the so-called *no-go theorems* by O'Raifeartaigh and McGlinn [3, 4] (see also [5]) long ago, which stated that there is no finite-dimensional Lie group containing the Poincaré group and any $SU(n)$ except for the direct product. It is worth mentioning that supersymmetry was originally introduced in the 70's by Salam and Strathdee [6] in an unsuccessful attemp to invalidate the no-go theorems.

However, the current skill in dealing with infinite-dimensional Lie groups tempts us into revisiting the question of the mixing of symmetries and, accordingly, the unification of interactions in terms of ordinary (though infinite-dimensional) Lie groups. In fact, there is an extremely simple, yet non-trivial, way of constructing a Lie group containing the Poincaré group and some unitary symmetry which accomplishes the above-mentioned task. This consists in looking at the $U(1)$ phase invariance of Quantum Theory as a 1-dimensional Cartan subgroup of a larger internal symmetry. Then, turning the space-time translation subgroup of the Poincaré group into a "local" group automatically promotes the original rigid internal symmetry to the gauge

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level in a non-direct product way. This of course entails a non-trivial mixing of gravity and the involved internal interaction associated with the given unitary symmetry.

In this letter, we shall approach the problem in the simplest and most economical way, in a Particle Mechanics (versus Field Theoretical) framework, leaving the more mathematically involved field formulation for the near future. To be precise, we face the situation that arises when promoting to the "local" level the space-time translations of the centrally extended space-time symmetry (either Galilei or Poincaré group), rather than the space-time symmetry itself.

The way of associating a physical dynamics with a specific symmetry can be accomplished by means of the rather standard co-adjoint orbits method of Kirillov [7], where the Lagrangian is seen as the local potential of the corresponding symplectic form, or through a generalized group approach to quantization which is directly related to the co-homological structure of the symmetry and leads directly to the quantum theory (see [8] and references there in).

To illustrate technically the present revisited Minimal Coupling Theory, let us consider the simpler case of the non-relativistic pure Lorentz force, keeping rigid the space-time translations. For this aim, we consider the Lie algebra $\tilde{\mathcal{G}}$ of the centrally extended Galilei group $\tilde{G}$ (only non-zero commutators):

$$
\begin{array}{rclrclrcl}
[X_{v^i},\, X_t] & = & X_{x^i} & [X_{v^i},\, X_{x^j}] & = & m\delta_{ij}X_\phi & & & \\
[X_{J^i},\, X_{J^j}] & = & \epsilon_{ij.}^{\;\;k}X_{J^k} & [X_{J^i},\, X_{v^j}] & = & \epsilon_{ij.}^{\;\;k}X_{v^k} & [X_{J^i},\, X_{x^j}] & = & \epsilon_{ij.}^{\;\;k}X_{x^k}
\end{array}
\tag{1}
$$

which leaves strictly invariant the extended Poincaré-Cartan form $\Theta = p_i dx^i - \frac{p^2}{2m} + d\phi$, $L_{X_a}\Theta = 0, \forall X_a \in \tilde{\mathcal{G}}$. This 1-form is defined on the extended phase space parametrized by (x^i, p_j, ϕ), where $e^{i\phi} \in U(1)$ is the phase transforming non-trivially under the Galilei group. It generalizes the Lagrangian and constitutes a potential for the symplectic form ω on the solution manifold (on trajectories $s(t)$, $(p_i dx^i - \frac{p^2}{2m}dt)|_{s(t)} = (p_i \dot{x}^i - \frac{p^2}{2m})|_{s(t)}dt)$.

Local $U(1)$ transformations generated by $f \otimes X_\phi$, f being a real function $f(\vec{x}, t)$, are incorporated into the scheme by adding to (1) the extra commutators [9]:

$$
[X_a,\, f \otimes X_\phi] = (L_{X_a} f) \otimes X_\phi \tag{2}
$$

Keeping the invariance of Θ under $f \otimes X_\phi$ requires modifying Θ by adding a connection piece $A = A_i dx^i + A_0 dt$ whose components transform under $U(1)(\vec{x}, t)$ as the space-time gradient of the function f.

The algebra (1)+(2) is infinite-dimensional but, if the functions f are real analytic, the dynamical content of it is addressed by the (co-homological) structure of the finite-dimensional subalgebra generated by $\tilde{\mathcal{G}}$ and the generators $f \otimes X_\phi$ with only linear functions. Thus, a very economical trick (eventually supported on unitarity grounds) for dealing with this sort of infinite-dimensional algebra consists in proceeding with the above mentioned 15-dimensional electromagnetic subgroup and then imposing the generic constraint $A^\mu = A^\mu(\vec{x}, t)$ on the symplectic structure. Let us call this group $\tilde{G}_E$, and the generators associated with linear functions in $f \otimes X_\phi$, X_{A^μ}.

The commutation relations of $\tilde{\mathcal{G}}_E$ are (we omit rotations, which operate in the standard way):

$$
\begin{array}{rclrclrcl}
[X_t,\, X_{x^i}] & = & 0 & [X_t,\, X_{v^i}] & = & -X_{x^i} & [X_{x^i}\,, X_{v^j}] & = & m\delta_{ij}X_\phi \\
[X_t,\, X_{A^i}] & = & 0 & [X_t,\, X_{A^0}] & = & -qX_\phi & [X_{x^i},\, X_{A^j}] & = & q\delta_{ij}X_\phi \\
[X_{x^i},\, X_{A^0}] & = & 0 & [X_{v^i},\, X_{A^i}] & = & \delta_{ij}X_{A^0} & [X_{v^i},\, X_{A^0}] & = & 0
\end{array}
\tag{3}
$$

where we have performed a new central extension parametrized by what proves to be the electric charge q.

The co-adjoint orbits of the group $\tilde{G}_E$ with non-zero electric charge have dimension 4+4 as a consequence of the Lie algebra cocycle piece $\Sigma(X_t,\, X_{A^0}) = -q$, which lends dynamical (symplectic) content to the time variable. This is a property inherited from the (centrally extended) conformal group from which $\tilde{\mathcal{G}}_E$ is an Inönü-Wigner contraction. In the case of the conformal group [10] the symplectic character of time is broken by means of a dynamical constraint (or by choosing a Poincaré vacuum) and the dimension 3+3 of the phase space is restored. Here the constraint $A^\mu = A^\mu(\vec{x}, t)$ also accomplishes this task at the same time as it introduces the notion of electromagnetic potential.

On a general orbit with non-zero q the extended Poincaré-Cartan form acquires the expression:

$$
\Theta = m\vec{v}\cdot d\vec{x} - \frac{1}{2}m\vec{v}^2 dt + q\vec{A}\cdot d\vec{x} - qA^0 dt + d\phi \tag{4}
$$

After imposing the above-mentioned constraint on A^μ, we compute the kernel of the presymplectic form $d\Theta$, i.e. the vector field (up to a multiplicative function) X such that $i_X d\Theta = 0$:

$$
X = \frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial\vec{x}} + \frac{q}{m}\left[\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)v^j - \frac{\partial A^0}{\partial x^i} - \frac{\partial A_i}{\partial t}\right]\frac{\partial}{\partial v_i} - [\frac{1}{2}m\vec{v}^2 + q(\vec{v}\cdot\vec{A} - A^0)]\frac{\partial}{\partial\phi}, \tag{5}
$$

where Latin indices are raised and lowered by the metric δ_{ij}. It defines the equations of motion of a charged particle moving in an electromagnetic field:

$$
\begin{array}{rcl}
\dfrac{d\vec{x}}{dt} & = & \vec{v} \\
m\dfrac{d\vec{v}}{dt} & = & q\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{A}) - \vec{\nabla}A^0 - \dfrac{\partial\vec{A}}{\partial t}\right],
\end{array}
\tag{6}
$$

which are nothing more than the standard Lorenz force equations. The same can be repeated with the centrally extended Poincaré group $\tilde{P}$ (see [10] and references therein) by promoting to "local" the $U(1)$ transformations and considering the finite-dimensional subgroup $\tilde{P}_E$ analogous to $\tilde{G}_E$.

Let us consider now the gravitational interaction. To this end, we start directly with the centrally extended Poincaré group $\tilde{P}$ and see how the fact that the translation generators produce the central term under commutation with some other generators (boosts) plays a singular role in the relationship between local space-time translations and local $U(1)$ transformations. Symbolically denoting the generators of translations by P, P_0, those of boosts by K and the central one by X_ϕ, we find:

$$
[K,\, f\otimes P] \simeq (L_K f)\otimes P + f\otimes(P_0 + X_\phi), \tag{7}
$$

which means that turning the translations into local symmetry entails also the local nature of the $U(1)$ phase. We expect, in this way, a non-trivial mixing of gravity and electromagnetism into an infinite-dimensional electro-gravitational group.

We shall follow identical steps as those given in the former example. The generators of local space-time translations associated with linear functions will be called $X_{h^{\mu\nu}}$, and the corresponding parameters $h^{\mu\nu}$ will also be constrained, in the form $h^{\mu\nu} = h^{\mu\nu}(\vec{x}, x^0)$, on the symplectic orbits. However, the co-homological structure of this finite-dimensional electro-gravitational subgroup, $\tilde{P}_{EG}$ is richer than that of $\tilde{P}_E$ and the exponentiation of the Lie algebra $\tilde{\mathcal{P}}_{EG}$ must be made, for the time being at least, order by order. Then, the explicit calculations will be kept up to order 3 in the group law. This will be enough to recognize the standard part of the interaction, i.e. the ordinary Lorentz force and the geodesic equations, although the latter in a quasi-linear approximation in terms of the metric $g^{\mu\nu} \equiv \eta^{\mu\nu} + h^{\mu\nu}$. But in addition, and associated with a new Lie algebra co-homology constant, κ, different from m and q and mixing both interactions, a new term appears in the Lorentz force made of the gravitational potential $h^{\mu\nu}$.

Let us write the algebra $\tilde{\mathcal{P}}_{EG}$ in an almost covariant way (the central extensions and induced deformations are necessarily non-covariant). To this end, we parametrize the Lorentz transformations with $L^{\mu\nu}$ as usual. The proposed explicit algebra is (we write for short $x_\mu,\ L_{\mu\nu}, ...$ instead of $X_{x^\mu},\ X_{L^{\mu\nu}}, ...$, and let Greek indices run from 0 to 3):

$$
\begin{aligned}
[x_\mu, L_{\nu\rho}] &= \eta_{\nu\mu} x_\rho - \eta_{\rho\mu} x_\nu + (m + \kappa q) c (\eta_{\rho\mu}\delta^0_\nu - \eta_{\nu\mu}\delta^0_\rho)\phi \\
[x_\mu, h_{\nu\rho}] &= \eta_{\nu\mu} x_\rho + \eta_{\rho\mu} x_\nu + mc(\eta_{\rho\mu}\delta^0_\nu + \eta_{\nu\mu}\delta^0_\rho)\phi \\
[x_\mu, A_\nu] &= q\eta_{\nu\mu}\phi \\
[L_{\mu\nu}, L_{\alpha\beta}] &= \eta_{\alpha\nu} L_{\mu\beta} - \eta_{\beta\nu} L_{\mu\alpha} - \eta_{\alpha\mu} L_{\nu\beta} + \eta_{\mu\beta} L_{\nu\alpha} \\
[L_{\mu\nu}, h_{\alpha\beta}] &= \eta_{\alpha\nu} h_{\mu\beta} + \eta_{\beta\nu} h_{\mu\alpha} - \eta_{\alpha\mu} h_{\nu\beta} - \eta_{\mu\beta} h_{\nu\alpha} - \\
&\quad \kappa c(\eta_{\alpha\nu}\delta^\rho_\beta\delta^0_\mu - \eta_{\mu\alpha}\delta^\rho_\beta\delta^0_\nu + \eta_{\nu\beta}\delta^\rho_\alpha\delta^0_\mu - \eta_{\mu\beta}\delta^\rho_\alpha\delta^0_\nu) A_\rho \\
[L_{\mu\nu}, A_\rho] &= \eta_{\rho\nu} A_\mu - \eta_{\rho\mu} A_\nu \\
[h_{\mu\nu}, h_{\alpha\beta}] &= \eta_{\alpha\nu} L_{\mu\beta} + \eta_{\beta\nu} L_{\mu\alpha} + \eta_{\alpha\mu} L_{\nu\beta} + \eta_{\mu\beta} L_{\nu\alpha} + \\
&\quad \kappa c \left[\eta_{\alpha\nu}\delta^{0\rho}_{\beta\mu} + \eta_{\beta\nu}\delta^{0\rho}_{\alpha\mu} + \eta_{\alpha\mu}\delta^{0\rho}_{\beta\nu} + \eta_{\beta\mu}\delta^{0\rho}_{\alpha\nu}\right] A_\rho \\
[h_{\mu\nu}, A_\rho] &= \eta_{\rho\nu} A_\mu + \eta_{\rho\mu} A_\nu
\end{aligned}
\qquad (8)
$$

where $\delta^{0\rho}_{\beta\mu} \equiv \delta^0_\beta\delta^\rho_\mu - \delta^0_\mu\delta^\rho_\beta$ is the Kronecker tensor.

It should be remarked that a consequence of having extended the Poincaré group prior to turning "local" space-time translations is the apearance of a term proportional (κ is a new co-homology constant) to A_ρ on the r.h.s. of the commutator $[L_{\mu\nu}, h_{\alpha\beta}]$, which will be responsible for a piece in the Lorentz force of gravitational origin. But even more, the term in A_ρ on the r.h.s of the commutator $[h_{\mu\nu}, h_{\alpha\beta}]$ could provide a "mixing vertex" at the Field Theory level.

We shall not dwell on explicit calculations in this letter and simply give the resulting equations of motion. Even more, we restrict ourselves to the "non-relativistic" limit stated by the Inönü-Wigner contraction with respect to the subgroup generated by (x_0, L_{ij}, A_k) (the standard $c \to \infty$ limit on the Poincaré group is an I-W contraction with respect to the subgroup (x_0, L_{ij})).

The contracted algebra reads:

$$
\begin{array}{rclrcl}
[x_0, L_{0i}] &=& x_i & [x_0, h_{00}] &=& -2m\phi \\
[x_0, h_{0i}] &=& x_i & [x_0, A_0] &=& q\phi \\
[x_i, L_{0j}] &=& -(m+\kappa q)\delta_{ij}\phi & [x_i, L_{jk}] &=& -\delta_{ij}x_k + \delta_{ik}x_j \\
[x_i, h_{0j}] &=& m\delta_{ij}\phi & [x_i, A_j] &=& -q\delta_{ij}\phi \\
[L_{0i}, L_{jk}] &=& -\delta_{ij}L_{0k} + \delta_{ki}L_{0j} & [L_{0i}, h_{0j}] &=& -\delta_{ij}h_{00} + \kappa\delta_{ij}A_0 \\
[L_{0i}, A_j] &=& -\delta_{ij}A_0 & [L_{ij}, L_{kl}] &=& -\delta_{kj}L_{il} + \delta_{lj}L_{ik} + \delta_{ik}L_{jl} - \delta_{il}L_{jk} \\
[L_{ij}, h_{0k}] &=& -\delta_{kj}h_{0i} + \delta_{ik}h_{0j} & [L_{ij}, h_{kl}] &=& -\delta_{kj}h_{il} - \delta_{lj}L_{ik} + \delta_{ik}L_{jl} + \delta_{il}L_{jk} \\
[L_{ij}, A_k] &=& -\delta_{kj}A_i + \delta_{ki}A_j & [h_{0i}, A_j] &=& -\delta_{ij}A_0
\end{array}
\tag{9}
$$

Writing $\vec{h}$ for (h^{0i}), we finally derive from this algebra the following Lagrangian and equations of motion, which at this contraction limit are indeed exact:

$$L = \frac{1}{2}(m+\kappa q)\,\dot{\vec{x}}^{\,2} - q(A^0 - \frac{\kappa}{8}\vec{h}^2) + q(\vec{A} - \frac{\kappa}{2}\vec{h})\cdot\dot{\vec{x}} + m(h^{00} + \frac{1}{4}\vec{h}^2) - m\vec{h}\cdot\dot{\vec{x}} \tag{10}$$

$$
\begin{aligned}
\frac{dx^i}{dt} &= v^i \ \ (\equiv L^{0i} + h^{0i}) \\
(m+\kappa q)\frac{d\vec{v}}{dt} &= q\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{A}) - \vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t}\right] \\
&- m\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{h}) - \vec{\nabla}h_{00} - \frac{\partial\vec{h}}{\partial t}\right] + \frac{m}{4}\nabla(\vec{h}\cdot\vec{h}) \\
&- \frac{\kappa q}{2}\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{h}) - \frac{1}{4}\nabla(\vec{h}\cdot\vec{h}) - \frac{\partial\vec{h}}{\partial t}\right]
\end{aligned}
\tag{11}
$$

The first three lines in (11) correspond to the standard motion of a particle in the presence of an electromagnetic and gravitational field (note that $h^{0i} = g^{0i} - \eta^{0i}$), except for the value of the inertial mass, which is corrected by κq, and within a quasi-linear approximation in the gravitational field. In fact, the third line contains one more order than the approximation in which the gravitational field looks like an electromagnetic one (standard gravito-electromagnetism [11]). The fourth, however, is quite new and represents another Lorentz-like force (proportional to q) generated by the gravitational potential and which must not be confused with the previous one. It is worth mentioning that the constant m in front of the term $\vec{\nabla}h_{00}$ in (11), naturally interpreted as a gravitational coupling, could acquire a different constant value, let us say g, allowed by the Lie algebra co-homology. Nevertheless, it must be made equal to m to recover the standard physics when switching the constant κ off. In this way, the equivalence principle between inertial and gravitational mass, in this co-homological setting, follows from the natural requirement of absence of a pathological mixing between electromagnetism and gravity when $\kappa = 0$.

It should also be noticed that in the standard formulation, and according to a reasoning not completely clear, the non-relativistic limit of the pure gravitational theory leads to just the term $\vec{\nabla}h^{00}$. Here, the non-relativistic limit, in general, appears as a clean Lie algebra contraction and permits forces derived from $\vec{h}$. As far as the magnitude of the new Lie algebra co-homology constant κ, it is limited by experimental clearance for the difference between particle and anti-particle mass, which for the electron is about $10^{-8}m_e$. Even though this is a small value, extremely dense rotating bodies could be able to produce measurable forces. In the other way around, a mixing of electromagnetism and gravity predicts a mass difference between charged particles and anti-particles, which could be experimentally tested.

Since the present theory has been formulated on symmetry grounds, it can be quantized on the basis of the group approach to quantization referred in [8]. Also, a natural yet highly non-elementary extension of the present theory to Quantum Field Theory is in course.

Finally, and as comented above, considering the group $U(1)$ as a Cartan subgroup of a larger internal symmetry group, for instance $SU(2) \otimes U(1)$ would result in additional phenomenology. Then, as in a QFT version, the production of Z_0 particles out of gravity might be permitted.

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[9] In general, $[f \otimes X_a, g \otimes X_b] = (f L_{X_a} g) \otimes X_b - (g L_{X_b} f) \otimes X_a + (fg) \otimes [X_a, X_b]$.

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